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BERNOULLIAN UTILITIES FOR MULTIPLE-FACTOR SITUATIONS

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BERNOULLIAN UTILITIES FOR MULTIPLE-FACTOR SITUATIONS

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13 ABSTRACT

Within the context of von Neumann-Morgenstern expected utility theory for n attributes, this paper presents axioms on preferences that are necessary and sufficient for the existence of a particular form of utility function that involves sums and product of real functions on the several attributes. For three attributes the form is $u(x_1,x_2,x_3) = u_1(x_1) + u_2(x_2) + u_3(x_3) + c_{12}\, f_1(x_1)f_2(x_2) + c_{13}\, f_1(x_1)f_3(x_3) + c_{23}\, f_2(x_2)f_3(x_3) + c_{123}\, f_1(x_1)f_2(x_2)f_3(x_3)$.

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BERNOULLIAN UTILITIES FOR MULTIPLE-FACTOR SITUATIONS

Peter C. Fishburn

III

1. INTRODUCTION

The von Neumann-Morgenstern expected-utility theory [2, 5, 6, 10, 14] specifies axioms--for a binary preference relation $\succ$ on the set P of simple probability distributions, or gambles p,q,..., on a set X of consequences x,y,...--that are necessary and sufficient for the existence of a real-valued utility function u on X which has the property that, for all $p,q \in P$,

$$p \succ q \text{ iff (if and only if) } E(u,p) > E(u,q), \tag{1}$$

where $E(u,p) =_{df} \Sigma\{p(x)u(x): \ x \in X\}$, the expected utility of gamble p.

Within twenty years after the initial (1944) publication of the von Neumann-Morgenstern axioms for Bernoullian expected utility (1), several investigators, motivated by the very practical concern of developing tractable techniques for analyzing complex risky decisions, began working out theories for special forms for the utility function u on X when X is equal to or is a subset of a product set $X_1 \times X_2 \times \ldots \times X_n$. One of the simplest forms for u in this multiple-factor setting is the additive form

$$u(x_1,\ldots,x_n) = u_1(x_1) + \ldots + u_n(x_n), \tag{2}$$

which was used extensively in nonrisky economic theory in the latter half of the ninteenth century. With $X = X_1 \times X_2 \times \ldots \times X_n$, a necessary and sufficient condition for (2) in the context of (1) was derived by Fishburn [1] and, independently, by Pollak [12]. Later, in [3], I note a necessary and sufficient condition for (2), given (1), when X is an arbitrary subset of $X_1 \times X_2 \times \ldots \times X_n$. Letting $\sim$ denote indifference, with $p \sim q$ iff not

$(p \succ q)$ & not $(q \succ p)$, this condition says that $p \sim q$ whenever, for each i, the marginal distributions on X_i derived from p and q are equal.

Somewhat more general forms for u in the context of (1) with $X = X_1 \times X_2 \times \ldots \times X_n$ have been axiomatized by Pollak [12], Raiffa [13], Keeney [7, 8, 9] and Meyer [11]. For $n = 2$ these forms include (2), $u(x_1,x_2) = f_1(x_1)f_2(x_2)$, $u(x_1,x_2) = u_1(x_1) + f_1(x_1)f_2(x_2)$ and $u(x_1,x_2) = u_2(x_2) + f_1(x_1)f_2(x_2)$. With n factors one encounters the additive and multiplicative forms along with related cases, including $u(x_1,\ldots,x_n) = \Sigma\{c_{i_1 \ldots i_r} f_{i_1}(x_{i_1}) \ldots f_{i_r}(x_{i_r}): 1 \leq r \leq n, 1 \leq i_1 < \ldots < i_r \leq n\}$ [9], which with $n = 3$ is $u(x_1,x_2,x_3) = c_1 f_1(x_1) + c_2 f_2(x_2) + c_3 f_3(x_3) + c_{12} f_1(x_1)f_2(x_2) + c_{13} f_1(x_1) f_3(x_3) + c_{23} f_2(x_2)f_3(x_3) + c_{123} f_1(x_1)f_2(x_2)f_3(x_3)$. The type of axiom used in the developments cited in this paragraph is as follows. Let $P[x_{i_1},\ldots,x_{i_r}]$ denote the subset of P in which the levels of $X_{i_1},\ldots,X_{i_r}$ are fixed at $x_{i_1},\ldots,x_{i_r}$ respectively, with $i_1 < \ldots < i_r$ and $r < n$. That is, $p \in P[x_{i_1},\ldots,x_{i_r}]$ iff the marginal of p on X_{i_s} assigns probability 1 to x_{i_s} for $s = 1,\ldots,r$. Then, for any two fixed $(x_{i_1},\ldots,x_{i_r})$ and $(y_{i_1},\ldots,y_{i_r})$, the restriction of $\succ$ on $P[y_{i_1},\ldots,y_{i_r}]$ results from the restriction of $\succ$ on $P[x_{i_1},\ldots,x_{i_r}]$ when $(x_{i_1},\ldots,x_{i_r})$ is replaced by $(y_{i_1},\ldots,y_{i_r})$. In other terms, this says that the decision maker's preference order on gambles defined on the product of a subset of the factors $X_1,\ldots,X_n$ with the levels of the other factors fixed (i.e., at $x_{i_1},\ldots,x_{i_r}$), does not depend on these fixed levels.

Recently [4], I examined an extension of these ideas in the two-factor case. This extension results in the form

$$u(x_1,x_2) = u_1(x_1) + u_2(x_2) + f_1(x_1)f_2(x_2), \qquad (3)$$

which incorporates the two-factor forms noted previously as special cases. In approximate terminology, (3) represents an additive form with independent multiplicative interaction: the functions u_1 and u_2 may be viewed as dealing with the main effects of x_1 and x_2; what is not accounted for by $u_1 + u_2$ is handled by the "residual" f_1f_2. Of course, if u_1 and u_2 in (3) can be made constant, then the so-called "residual" f_1f_2 tells the whole story with u taking the straight multiplicative form.

The main purpose of the present paper is to extend (3) to n factors. In doing this it will be assumed throughout that (1) holds with $X = X_1 \times X_2 \times \ldots \times X_n$. The particular extension of (3) that is obtained is

$$u(x_1,\ldots,x_n) = \sum_{i=1}^{n} u_i(x_i) + \Sigma\{c_{i_1\ldots i_r} f_{i_1}(x_{i_1})\ldots f_{i_r}(x_{i_r}):\ 2 \leq r \leq n \text{ and } 1 \leq i_1 < \ldots < i_r \leq n\} \tag{4}$$

where the c's are constants and u_i and f_i are real-valued functions on X_i. For n = 3, (4) is

$$\begin{aligned} u(x_1,x_2,x_3) = u_1(x_1) + u_2(x_2) + u_3(x_3) + c_{12}f_1(x_1)f_2(x_2) + c_{13}f_1(x_1)f_3(x_3) \\ + c_{23}f_2(x_2)f_3(x_3) + c_{123}f_1(x_1)f_2(x_2)f_3(x_3). \end{aligned}$$

Although this form is more complex than some others discussed previously, it is still tractable from an estimation (scaling) and analysis viewpoint for smaller n. In particular, it requires estimation of two univariate functions for each i plus the constant c's.

The next section presents a two-period income-stream example (n = 2) to illustrate the potential applicability of (3) when simpler forms cannot

be used for u (although such simpler forms might yield "acceptable" numerical approximations for analysis). The general theory for (4) is in section 3. Section 4 contains the sufficiency proof of the main theorem, and the final section discusses aspects of scaling (estimation) procedures for the functions and constants in (4).

2. EXAMPLE

This section illustrates (3) with a two-period income example. It is supposed that the individual will receive an income x_1 between \$10000 and \$30000 at the beginning of the next year (period 1), and that he will receive an income x_2 between \$10000 and \$30000 at the beginning of the year after that (period 2). His preferences for the extreme combinations are

$$(\$30000,\$30000) \succ (\$30000,\$10000) \succ (\$10000,\$30000) \succ (\$10000,\$10000),$$

which shows that if he could have \$30000 in one period and \$10000 in the other then he would rather have the \$30000 in the first period.

Using a typical scaling procedure for Bernoullian utilities he estimates that

$$(\$30000,\$10000) \sim [(\$30000,\$30000) \text{ with pr. } \tfrac{3}{4} \text{ or } (\$10000,\$10000) \text{ with pr. } \tfrac{1}{4}]$$
$$(\$10000,\$30000) \sim [(\$30000,\$30000) \text{ with pr. } \tfrac{2}{3} \text{ or } (\$10000,\$10000) \text{ with pr. } \tfrac{1}{3}].$$

Given \$10000 for sure in period 1, he is risk-averse over x_2 in period 2, but given \$30000 in period 1 he is risk-neutral over x_2 in period 2. Likewise, with a guarantee of \$10000 in period 2, he is risk-averse over x_1 in period 1, but given \$30000 in period 2 he is risk-neutral over x_1 in period 1.

In specifying a utility function on [\$10000,\$30000] × [\$10000,\$30000] which has these properties and has the form shown in (3), we use the following linear transformations for notational convenience:

$$\alpha = (x_1 - \$10000)/\$20000 \qquad \beta = (x_2 - \$10000)/\$20000.$$

For analytical convenience we assume that u on the (α,β) pairs in $[0,1] \times [0,1]$ is given by

$$u(\alpha,\beta) = 1.8\alpha^{1/2} + 1.6\beta^{1/2} - [1.8\alpha^{1/2} - .8\alpha][1.6\beta^{1/2} - .6\beta]. \qquad (5)$$

This has the form of (3) and gives $u(1,1) = 2.4$, $u(1,0) = 1.8$, $u(0,1) = 1.6$ and $u(0,0) = 0$, which satisfies the two indifference expressions written above since $u(1,0) = \frac{3}{4}u(1,1) + \frac{1}{4}u(0,0)$ and $u(0,1) = \frac{2}{3}u(1,1) + \frac{1}{3}u(0,0)$.

Figure 1 shows conditional utility curves for α in period 1, given fixed β in period 2, and conditional utility curves for β in period 2, given fixed α in period 1. In the period 1 diagram the lowest curve is $u(\alpha,0) = 1.8\alpha^{1/2}$ for $\beta = 0$ [\$10000 in period 2], and the highest curve is $u(\alpha,1) = 1.6 + .8\alpha$

<u>Figure 1 about here</u>

for $\beta = 1$ [\$30000 in period 2]. As one progresses from the lowest period 1 curve to the highest, the period 1 conditional functions becomes increasingly less risk-averse, with the highest curve being risk-neutral. Similar remarks apply to the period 2 curves.

The main point of this example is that the intuitively-reasonable conditional utility curves in the two pictures of Figure 1 cannot be generated by any of the specializations of (3) mentioned in section 1 (i.e., additive,

multiplicative, $u_1 + f_1 f_2$, and $u_2 + f_1 f_2$), whereas they are in fact generated by (3) in the specific form of (5). Thus (3), or its extension (4) to n factors, permits more realistic characterizations of multiple-factor utility functions in terms of combinations of functions on the individual factors.

We now turn to the theory behind (4).

3. THEORY

Throughout, it is assumed that (1) holds with $X = X_1 \times X_2 \times \ldots \times X_n$. In developing necessary and sufficient conditions for (4) we shall let P_i be the set of simple probability distributions on X_i, with members $p_i, q_i, \ldots$. In addition, x^i denotes an $n - 1$ tuple $(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n)$ in the set $X^i = X_1 \times \ldots \times X_{i-1} \times X_{i+1} \times \ldots \times X_n$. The pair (x^i, p_i) in $X^i \times P_i$ can also be viewed as a gamble $p \in P$ whose marginal for X_i is p_i and which has marginals for the X_j for $j \neq i$ which assign probability 1 to the x^i_j. That is, (x^i, p_i) is in $P[x^i]$.

In Keeney's terms [9], X_i is "utility independent" of X^i iff $[(x^i, p_i) \succ (x^i, q_i)$ iff $(y^i, p_i) \succ (y^i, q_i)]$ for all $x^i, y^i \in X^i$ and $p_i, q_i \in P_i$. This says that the order on P_i conditional on a fixed x^i does not depend on the fixed x^i. When X_i is "utility independent" of X^i for $i = 1, \ldots, n$, Keeney notes [9, p. 284] that u on X takes the form $u(x_1, \ldots, x_n) = \Sigma\{c_{i_1 \ldots i_r} f_{i_1}(x_{i_1}) \ldots f_{i_r}(x_{i_r}):\ 1 \leq r \leq n,\ 1 \leq i_1 < \ldots < i_r \leq n\}$, which is obtainable from (4) when u_i therein is replaced by $c_i f_i$.

To obtain the full form of (4), in which the u_i can be very different than the f_i, we require a generalization of the "utility independent" idea.

This generalization is accomplished by using two, rather than one, conditioning values of X^i in a 50-50 lottery formulation. Letting $\frac{1}{2}p + \frac{1}{2}q$ represent the gamble with probability $\frac{1}{2}$ for p and probability $\frac{1}{2}$ for q (the one of p and q that results will then be played), or equivalently the distribution in P that assigns probability $\frac{1}{2}p(x) + \frac{1}{2}q(x)$ to each $x \in X$, we define the doubly-conditioned order $\succ_{x^i y^i}$ on P_i by

$$p_i \succ_{x^i y^i} q_i \quad \text{iff} \quad \frac{1}{2}(x^i,p_i) + \frac{1}{2}(y^i,q_i) \succ \frac{1}{2}(x^i,q_i) + \frac{1}{2}(y^i,p_i). \tag{6}$$

This can be visualized by a standard array of two compound gambles p and q as follows:

	Heads	Tails
p	(x^i,p_i)	(y^i,q_i)
q	(x^i,q_i)	(y^i,p_i)

Presuming that Heads and Tails are believed to be equally likely, p represents $\frac{1}{2}(x^i,p_i) + \frac{1}{2}(y^i,q_i)$, which gives (x^i,p_i) if Heads obtains and (y^i,q_i) if Tails obtains. Likewise, q represents $\frac{1}{2}(x^i,q_i) + \frac{1}{2}(y^i,p_i)$, which is the same as p except that p_i and q_i have been interchanged. Definition (6) says that $p_i \succ_{x^i y^i} q_i$ if and only if the first row of the array is preferred to the second row, or if $p \succ q$. If indifference $p \sim q$ should hold for all possible arrays of the foregoing form $(i = 1,\ldots,n)$, then u has the additive form (2) [1].

According to the definition, $p_i \succ_{x^i y^i} q_i$ iff $q_i \succ_{y^i x^i} p_i$, so that $\succ_{y^i x^i}$ is the converse or dual or $\succ_{x^i y^i}$. (In terms of the array, interchange

the two columns in the matrix.) In addition, with ϕ the empty set, $\succ_{x^i x^i} = \phi$ for all $x^i \in X^i$. That is, if y^i in the array or in (6) is replaced by x^i, then $p = q$ so that $p \sim q$, or $p_i \sim_{x^i x^i} q_i$, for all $p_i, q_i \in P_i$.

To expose the desired axioms for (4) on the basis of the $\succ_{x^i y^i}$, we work momentarily with $i = 1$. Suppose (4) holds and there exist $p_1, q_1 \in P_1$ such that $p_1 \succ_{x^1 y^1} q_1$ for some $x^1, y^1 \in X^1$. Then, using (1) and (4) on the $\succ$ expression of (6) we obtain

$$[f_1(p_1) - f_1(q_1)][\Sigma\{c_{1 i_1 \ldots i_r}[f_{i_1}(x^1_{i_1})\ldots f_{i_r}(x^1_{i_r}) - f_{i_1}(y^1_{i_1})\ldots f_{i_r}(y^1_{i_r})]: \\ 1 \le r \le n-1,\ 2 \le i_1 < \ldots < i_r \le n\}] > 0,$$

where $f_1(p_1) = \Sigma p_1(x_1) f_1(x_1)$ and similarly for $f_1(q_1)$. The important aspect of this expression is that the terms in f_1 are separated from the terms in the other f_i. Letting $g(x^1, y^1)$ represent the bracketted sum in the inequality, we rewrite it as

$$[f_1(p_1) - f_1(q_1)]g(x^1, y^1) > 0.$$

Because of the way this was developed, we have

$$p_1 \succ_{x^1 y^1} q_1 \quad \text{iff} \quad [f_1(p_1) - f_1(q_1)]g(x^1, y^1) > 0,$$

for all $p_1, q_1 \in P_1$. Now when x^1 here is replaced by some other $z^1 \in X^1$, exactly one of the following three things must occur:

1. $g(z^1, y^1)$ has the same sign as $g(x^1, y^1)$: then it must be true, by a simple analysis of signs, that $\succ_{z^1 y^1} = \succ_{x^1 y^1}$;

2. $g(z^1,y^1)$ has the opposite sign from the sign of $g(x^1,y^1)$: then it must be true that $\succ_{z^1y^1}$ is the converse or dual of $\succ_{x^1y^1}$, or $\succ_{z^1y^1} = \succ_{y^1x^1}$;

3. $g(z^1,y^1)$ equals zero, in which case $p_1 \sim_{z^1y^1} q_1$ for all $p_1,q_1 \in P_1$, so that $\succ_{z^1y^1} = \phi$.

To summarize: if (4) holds and if $p_1 \succ_{x^1y^1} q_1$ for some $p_1,q_1 \in P_1$ then, for every $z^1 \in X^1$, $\succ_{z^1y^1} \in \{\succ_{x^1y^1}, \succ_{y^1x^1}, \phi\}$. A similar conclusion arises when $i = 1$ is replaced by any $i > 1$.

AXIOM 1.i. <u>If</u> $\succ_{x^iy^i} \neq \phi$ <u>for some</u> $x^i,y^i \in X^i$, <u>then for some</u> $\succ_{x^iy^i}$ <u>on</u> P_i <u>which is nonempty</u>, $\succ_{z^iy^i} \in \{\succ_{x^iy^i}, \succ_{y^ix^i}, \phi\}$ <u>for each</u> $z^i \in X^i$.

As just demonstrated, Axiom 1.i is necessary for (4), for $i = 1,\ldots,n$. As the following theorem states, Axioms 1.1,...,1.n together are sufficient for (4). The proof of sufficiency is given in the next section.

THEOREM 1. <u>There exist real-valued functions</u> u_i <u>and</u> f_i <u>on</u> X_i <u>for</u> $i = 1,\ldots,n$ <u>and constants</u> $c_{i_1\ldots i_r}$ <u>which satisfy</u> (4) <u>for all</u> $x \in X$ <u>if, and only if</u>, <u>Axiom</u> 1.i <u>holds for</u> $i = 1,\ldots,n$.

4. PROOF

For the sufficiency proof of Theorem 1, assume that Axioms 1.1,...,1.n hold. Then, according to [4], there exist real-valued functions u_i and f_i on X_i, and real-valued functions v_i and g_i on X^i, such that, for all $x \in X$,

$$u(x) = u_i(x_i) + v_i(x^i) + f_i(x_i)g_i(x^i) \qquad i = 1,\ldots,n. \tag{7}$$

Without loss in generality (but perhaps involving an origin shift for u), we can suppose that there exists an $x^0 = (x_1^0,\ldots,x_n^0)$ in X such that all functions in (7) equal zero at x^0. (See [4].)

If f_i in (7) is multiplied by any nonzero constant and g_i is multiplied by the reciprocal of this constant then (7) is unchanged. Moreover, if (7) is additive for some i in the sense that $u(x) = u_i(x_i) + v_i(x^i)$, then we can set $g_i \equiv 0$ and define f_i in any way we please with $f_i(x_i^0) = 0$. Hence, in any event we can select an $x_i^* \in X_i$ for each i and require that $f_i(x_i^*) = 1$, assuming of course that X_i has at least two elements. (If X_i were a singleton it would presumably be omitted from the product set X or incorporated into some other X_j.)

To arrive at (4) from (7) we proceed through the equations in (7) in a systematic manner, beginning with i = 1,2, then adding the next i in each new step.

Using (7) with i = 1 and i = 2 and setting $x_1 = x_1^0$ we get

$$v_1(x_2,\ldots,x_n) = u_2(x_2) + v_2(x_1^0,x_3,\ldots,x_n) + f_2(x_2)g_2(x_1^0,x_3,\ldots,x_n).$$

Likewise, on setting $x_1 = x_1^*$ we get $u_1(x_1^*) + v_1(x_2,\ldots,x_n) + g_1(x_2,\ldots,x_n) = u_2(x_2) + v_2(x_1^*,x_3,\ldots,x_n) + f_2(x_2)g_2(x_1^*,x_3,\ldots,x_n)$, which on substitution for v_1 as displayed above gives

$$\begin{aligned} g_1(x_2,\ldots,x_n) = {} & v_2(x_1^*,x_3,\ldots,x_n) - v_2(x_1^0,x_3,\ldots,x_n) \\ & + f_2(x_2)[g_2(x_1^*,x_3,\ldots,x_n) - g_2(x_1^0,x_3,\ldots,x_n)] - u_1(x_1^*). \end{aligned}$$

Thus, in these expressions for v_1 and g_1, x_1 has been effectively nullified (by fixing it at x_1^0 or x_1^*) and x_2 has been "separated" from $(x_3,\ldots,x_n)$.

Replacement in $u(x) = u_1(x_1) + v_1(x_2,\ldots,x_n) + f_1(x_1)g_1(x_2,\ldots,x_n)$ of the foregoing expressions for v_1 and g_1 is the first step in proceeding towards (4).

The next step is to add in (7) for $i = 3$ in breaking down the v_2 and g_2 expressions on the right sides of the equations given above for v_1 and g_1. (It may be unclear at this point as to the fate of $-u_1(x_1^*)$ at the end of the g_1 expression. In point of fact, $u_1(x_1^*) = u(x_1^*,x_2^0,\ldots,x_n^0)$, and it cancels out in the final step of the proof.) With $a \in \{0,*\}$, the equivalence of (7) for $i = 2,3$ with x_2 set at x_2^0 therein gives

$$\begin{aligned} v_2(x_1^a,x_3,\ldots,x_n) = u_3(x_3) &+ v_3(x_1^a,x_2^0,x_4,\ldots,x_n) \\ &+ f_3(x_3)g_3(x_1^a,x_2^0,x_4,\ldots,x_n). \end{aligned}$$

Subtracting this for $a = 0$ from $a = *$ for substitution in g_1 gives $v_2(x_1^*,x_3,\ldots,x_n) - v_2(x_1^0,x_3,\ldots,x_n) = v_3(x_1^*,x_2^0,x_4,\ldots,x_n) - v_3(x_1^0,x_2^0,x_4,\ldots,x_n) + f_3(x_3)[g_3(x_1^*,x_2^0,x_4,\ldots,x_n) - g_3(x_1^0,x_2^0,x_4,\ldots,x_n)]$. Using $i = 2,3$ in (7) with x_2 set at x_2^*, and using the foregoing expression for $v_2(x_1^a, x_3,\ldots,x_n)$, we obtain

$$\begin{aligned} g_2(x_1^a,x_3,\ldots,x_n) = {} & v_3(x_1^a,x_2^*,x_4,\ldots,x_n) - v_3(x_1^a,x_2^0,x_4,\ldots,x_n) \\ & + f_3(x_3)[g_3(x_1^a,x_2^*,x_4,\ldots,x_n) - g_3(x_1^a,x_2^0,x_4,\ldots,x_n)] \\ & - u_2(x_2^*). \end{aligned}$$

The general pattern should be clear at this point. When we introduce expression i from (7) into the process, we get

$$v_{i-1}(x_1^{a_1},\ldots,x_{i-2}^{a_{i-2}},x_i,\ldots,x_n) = u_i(x_i) + v_i(x_1^{a_1},\ldots,x_{i-2}^{a_{i-2}},x_{i-1}^0,x_{i+1},\ldots,x_n)$$
$$+ f_i(x_i)g_i(x_1^{a_1},\ldots,x_{i-2}^{a_{i-2}},x_{i-1}^0,x_{i+1},\ldots,x_n)$$

$$g_{i-1}(x_1^{a_1},\ldots,x_{i-2}^{a_{i-2}},x_i,\ldots,x_n) = v_i(x_1^{a_1},\ldots,x_{i-2}^{a_{i-2}},x_{i-1}^*,x_{i+1},\ldots,x_n)$$
$$- v_i(x_1^{a_1},\ldots,x_{i-2}^{a_{i-2}},x_{i-1}^0,x_{i+1},\ldots,x_n)$$
$$+ f_i(x_i)[g_i(x_1^{a_1},\ldots,x_{i-2}^{a_{i-2}},x_{i-1}^*,x_{i+1},\ldots,x_n)$$
$$- g_i(x_1^{a_1},\ldots,x_{i-2}^{a_{i-2}},x_{i-1}^0,x_{i+1},\ldots,x_n)]$$
$$- u_{i-1}(x_{i-1}^*),$$

where $a_j \in \{0,*\}$ for each $j \leq i - 2$. This continues through $i = n$, at which point we have

$$v_{n-1}(x_1^{a_1},\ldots,x_{n-2}^{a_{n-2}},x_n) = u_n(x_n) + v_n(x_1^{a_1},\ldots,x_{n-2}^{a_{n-2}},x_{n-1}^0)$$
$$+ f_n(x_n)g_n(x_1^{a_1},\ldots,x_{n-2}^{a_{n-2}},x_{n-1}^0)$$

$$g_{n-1}(x_1^{a_1},\ldots,x_{n-2}^{a_{n-2}},x_n) = v_n(x_1^{a_1},\ldots,x_{n-2}^{a_{n-2}},x_{n-1}^*) - v_n(x_1^{a_1},\ldots,x_{n-2}^{a_{n-2}},x_{n-1}^0)$$
$$+ f_n(x_n)[g_n(x_1^{a_1},\ldots,x_{n-2}^{a_{n-2}},x_{n-1}^*) - g_n(x_1^{a_1},\ldots,$$
$$x_{n-2}^{a_{n-2}},x_{n-1}^0)]$$
$$- u_{n-1}(x_{n-1}^*).$$

Using (7) with $i = n$, the v_n and g_n terms on the right of these expressions are replaced with definite u values as follows:

$$v_n(x_1^{a_1},\ldots,x_{n-1}^{a_{n-1}}) = u(x_1^{a_1},\ldots,x_{n-1}^{a_{n-1}},x_n^0) \tag{8}$$

$$g_n(x_1^{a_1},\ldots,x_{n-1}^{a_{n-1}}) = u(x_1^{a_1},\ldots,x_{n-1}^{a_{n-1}},x_n^*) - u(x_1^{a_1},\ldots,x_{n-1}^{a_{n-1}},x_n^0)$$
$$- u_n(x_n^*). \tag{9}$$

Beginning with $u(x) = u_1(x_1) + v_1(x') + f_1(x_1)g_1(x^1) = u_1(x_1) + [u_2(x_2) + v_2(x_1^0,x_3,\ldots,x_n) + f_2(x_2)g_2(x_1^0,x_3,\ldots,x_n)] + f_1(x_1)[v_2(x_1^*,x_3,\ldots,x_n) - v_2(x_1^0,x_3,\ldots,x_n) + f_2(x_2)[g_2(x_1^*,x_3,\ldots,x_n) - g_2(x_1^0,x_3,\ldots,x_n)] - u_1(x_1^*)] = \ldots$, induction on k shows that (with * replaced by 1 for analytical convenience)

$$\begin{aligned} u(x) = {} & \sum_{i=1}^{k} u_i(x_i) + v_k(x_1^0,\ldots,x_{k-1}^0,x_{k+1},\ldots,x_n) - \sum_{i=1}^{k-1} f_i(x_i)u_i(x_i^1) \\ & + \Sigma\{f_{i_1}(x_{i_1})\ldots f_{i_r}(x_{i_r})[\Sigma\{(-1)^{r+\Sigma a_i} v_k(x_1^{a_1},\ldots,x_{k-1}^{a_{k-1}},x_{k+1},\ldots,x_n): \\ & \quad a_i = 0 \text{ if } i \notin \{i_1,\ldots,i_r\},\ a_i \in \{0,1\} \text{ if } i \in \{i_1,\ldots,i_r\}\}]: \\ & \quad 1 \le r < k,\ 1 \le i_1 <\ldots< i_r < k\} \\ & + f_k(x_k)[g_k(x_1^0,\ldots,x_{k-1}^0,x_{k+1},\ldots,x_n) + \Sigma\{f_{i_1}(x_{i_1})\ldots f_{i_r}(x_{i_r}) \\ & \quad [\Sigma\{(-1)^{r+\Sigma a_i} g_k(x_1^{a_1},\ldots,x_{k-1}^{a_{k-1}},x_{k+1},\ldots,x_n): \ a_i = 0 \text{ if} \\ & \quad i \notin \{i_1,\ldots,i_r\},\ a_i \in \{0,1\} \text{ if } i \in \{i_1,\ldots,i_r\}\}]: \ 1 \le r < k, \\ & \quad 1 \le i_1 <\ldots< i_r < k\}]. \end{aligned}$$

Setting k = n here and using (8) and (9) we get

$$\begin{aligned} u(x) = {} & \sum_{i=1}^{n} u_i(x_i) - \sum_{i=1}^{n-1} f_i(x_i)u_i(x_i^1) + \Sigma\{f_{i_1}(x_{i_1})\ldots f_{i_r}(x_{i_r})[\Sigma\{(-1)^{r+\Sigma a_i} \\ & \quad u(x_1^{a_1},\ldots,x_{n-1}^{a_{n-1}},x_n^0): \ a_i = 0 \text{ if } i \notin \{i_1,\ldots,i_r\},\ a_i \in \{0,1\} \\ & \quad \text{otherwise}\}]: \ 1 \le r < n,\ 1 \le i_1 <\ldots< i_r < n\} \end{aligned}$$

$$+ f_n(x_n)[\Sigma\{f_{i_1}(x_{i_1})\dots f_{i_r}(x_{i_r})[\Sigma\{(-1)^{r+\Sigma a_i}[u(x_1^{a_1},\dots,x_{n-1}^{a_{n-1}},x_n^1) -$$

$$u(x_1^{a_1},\dots,x_{n-1}^{a_{n-1}},x_n^0) - u_n(x_n^1)]:\ a_i = 0 \text{ if } i \notin \{i_1,\dots,i_r\},$$

$$a_i \in \{0,1\} \text{ otherwise}\}]:\ 1 \le r < n,\ 1 \le i_1 <\dots< i_r < n\}].$$

The third main sum here with $r = 1$ and $i = i_r < n$ gives the subterm $f_i(x_i)u(x_1^0,\dots,x_{i-1}^0,x_i^1,x_{i+1}^0,\dots,x_n^0)$, which cancels with $- f_i(x_i)u_i(x_i^1)$ from the second main sum. Within the last main sum, we find the expression $- \Sigma\{(-1)^{r+\Sigma a_i}u_n(x_n^1):\ a_i = 0 \text{ if } i \notin \{i_1,\dots,i_r\},\ a_i \in \{0,1\} \text{ otherwise}\} = - u_n(x_n^1)(-1)^r\, \Sigma\{(-1)^{\Sigma a_i}:\dots\}$. This equals zero since $\Sigma\{(-1)^{\Sigma a_i}:\dots\} = 0$. What remains then in the above form for $u(x)$ is precisely (4), with $c_{i_1\dots i_r}$ in (4) specified by

$$c_{i_1\dots i_r} = \Sigma\{(-1)^{r+\Sigma a_i}u(x_1^{a_1},\dots,x_n^{a_n}):\ a_i = 0 \text{ if}$$

$$i \notin \{i_1,\dots,i_r\},\ a_i \in \{0,1\} \text{ otherwise}\}. \qquad (10.$$

5. SCALING

Not only does the preceding proof establish Theorem 1, but it shows how one can determine the u_i and f_i functions along with the constants $c_{i_1\dots i_r}$ in (4). The purpose of this section is to clarify aspects of the scaling procedure. As in the latter part of the preceding section, we use the superscript 1 in the sense of $x^1 = (x_1^1,\dots,x_n^1)$ in X rather than to denote an element in X^1.

As in the proof, we fix an element $x^0 \in X$ and require all functions, including u, to equal zero at x^0. This is quite permissible and fixes an origin for each function. Using (7) it follows that

$$u_i(x_i) = u(x_1^0,\ldots,x_{i-1}^0,x_i,x_{i+1}^0,\ldots,x_n^0)$$

$$v_i(x_1,\ldots,x_{i-1},x_{i+1},\ldots,x_n) = u(x_1,\ldots,x_{i-1},x_i^0,x_{i+1},\ldots,x_n).$$

In considering f_i we distinguish two cases. The first of these arises when (7) is additive in i in the sense that

$$u(x) = u_i(x_i) + v_i(x_1,\ldots,x_{i-1},x_{i+1},\ldots,x_n). \tag{11}$$

Although we used an artifice of defining $f_i(x_i^*) = f_i(x_i^1) = 1$ in this case in the preceding proof, it is quite all right to take $f_i \equiv 0$ when (11) holds. In fact, f_i can be defined in any way here since, as is easily seen with the use of (11), every $c_{i_1\ldots i_r}$ for (4), as specified by (10), which includes i among $i_1\ldots i_r$ equals zero. Hence f terms in (4) which include f_i vanish from (4) when (11) holds. Moreover, we need not worry about selecting an $x_i^1 = x_i^*$ element in this case for estimating the nonzero c coefficients by (10).

To summarize the first case: when (11) holds, or equivalently when $\succ_{x^i y^i} = \phi$ for all $x^i,y^i \in X^i$, only u_i as specified above needs to be estimated.

The second case arises when (11) is false. In this case there is an $(x_1^{(i)},\ldots,x_{i-1}^{(i)},x_i^1,x_{i+1}^{(i)},\ldots,x_n^{(i)})$ in X such that

$$\begin{aligned} f_i(x_i^1)g_i(x_1^{(i)},\ldots,x_{i-1}^{(i)},x_{i+1}^{(i)},\ldots,x_n^{(i)}) &= u(x_1^{(i)},\ldots,x_i^1,\ldots,x_n^{(i)}) \\ &- u(x_1^0,\ldots,x_i^1,\ldots,x_n^0) - u(x_1^{(i)},\ldots,x_i^0,\ldots,x_n^{(i)}) \neq 0. \end{aligned} \tag{12}$$

Thus the g_i term is nonzero. Dividing both sides by this term and replacing x_i^1 by x_i we have

$$f_i(x_i) = \lambda_i[u(x_1^{(i)},\ldots,x_i,\ldots,x_n^{(i)}) - u_i(x_i) - u(x_1^{(i)},\ldots,x_i^0,\ldots,x_n^{(i)})]$$

where, in line with the preceding proof, λ_i is determined by the requirement that $f_i(x_i^1) = 1$:

$$\lambda_i = [u(x_1^{(i)},\ldots,x_i^1,\ldots,x_n^{(i)}) - u(x_1^0,\ldots,x_i^1,\ldots,x_n^0) - u(x_1^{(i)},\ldots,x_i^0,\ldots,x_n^{(i)})]^{-1}.$$

Finally, the $c_{i_1 \ldots i_r}$ coefficients for (4) for those $i_1 \ldots i_r$ that includes no i for which (11) holds, are determined by (10) using x^0 and the x_i^1 as defined in this paragraph.

If there are exactly k values of i for which (11) is false [k cannot equal 1, for if n - 1 of the i satisfy (11) then the other i satisfies (11) also] then, in addition to the n u_i functions we need to estimate k f_i functions, each of which requires determination of $u(x_1^{(i)},\ldots,x_{i-1}^{(i)},x_i, x_{i+1}^{(i)},\ldots,x_n^{(i)})$ in addition to $u_i(x_i)$. To complete the specification of f_i, the constant values $u(x_1^{(i)},\ldots,x_i^1,\ldots,x_n^{(i)})$ and $u(x_1^{(i)},\ldots,x_i^0,\ldots,x_n^{(i)})$ are required along with $u(x_1^0,\ldots,x_i^1,\ldots,x_n^0)$, which will be one of the 2^k $u(x_1^{a_1},\ldots,x_n^{a_n})$ values required for the $c_{i_1 \ldots i_r}$ in (10).

The estimation procedure simplies somewhat if for each of the k i values for which (11) fails it is possible to select the x_i^1 element in such a way that, with $I = \{i\text{: (11) is false for } i\}$, the right side of (12) is nonzero when $x_j^{(i)} = x_j^1$ for each $j \in I - \{i\}$ and $x_j^{(i)} = x_j^0$ for each $j \notin I$. For example, if this can be done when k = n (i.e., when (7) is "additive" for no i) we get

$$u_i(x_i) = u(x_1^0,\ldots,x_{i-1}^0,x_i,x_{i+1}^0,\ldots,x_n^0)$$

$$f_i(x_i) = \lambda_i[u(x_1^1,\ldots,x_{i-1}^1,x_i,x_{i+1}^1,\ldots,x_n^1) - u_i(x_i) - u(x_1^1,\ldots,x_{i-1}^1,x_i^0,x_{i+1}^1,\ldots,x_n^1)]$$

$$\lambda_i = [u(x^1) - u(x_1^0,\ldots,x_i^1,\ldots,x_n^0) - u(x_1^1,\ldots,x_i^0,\ldots,x_n^1)]^{-1}$$

with the $c_{i_1 \ldots i_r}$ determined from (10) using x^0 and x^1. In this situation we need to estimate the two univariate functions $u(x_1^0,\ldots,x_i,\ldots,x_n^0)$ and $u(x_1^1,\ldots,x_i,\ldots,x_n^1)$ for each i along with the 2^n values of $u(x_1^{a_1},\ldots,x_n^{a_n})$ [with $u(x^0) = 0$].

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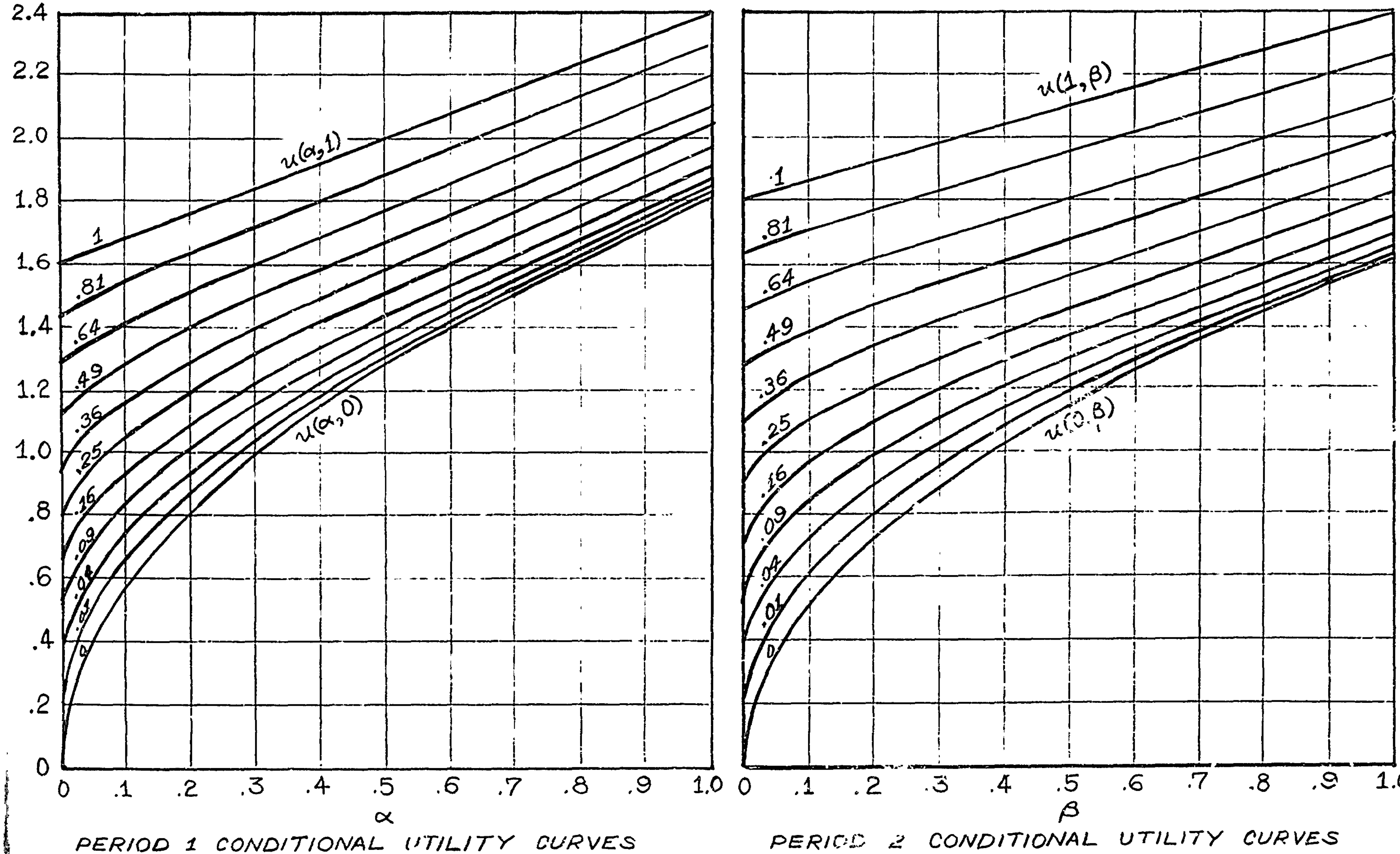
u(α,1)
u(α,0)
1
.81
.64
.49
.36
.25
.16
.09
.04
.01
0
α
PERIOD 1 CONDITIONAL UTILITY CURVES
u(1,β)
u(0,β)
β
PERIOD 2 CONDITIONAL UTILITY CURVES

FIGURE 1